UNCERTAINTIES TUTORIAL

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1 INTRODUCTION

For the past thirty years or so, much has been written about the uncertainty of measurements. During this period, a consensus has developed concerning pragmatic, and adequate, methods of evaluating and expressing the uncertainty of a measurement. As a result of this development, the International Organisation for Standardisation (ISO), in 1993, published "Guide to the Expression of Uncertainty in Measurement" [1] with the endorsement of six other international organisations (BIPM/OIML/IEC/IUPAP/IUPAC/IFCC). Since then, further guidance documents have been issued (and continue to be issued), based on the ISO Guide, for specific applications. The material in these tutorial notes is also in-line with these internationally accepted procedures.

These notes begin with a review of the fundamental concepts involved in uncertainty evaluation techniques. This is followed by an example demonstrating the application of these concepts to a typical measurement. The example, taken from a paper presented at the sixth British Electromagnetic Measurements Conference held at NPL Teddington in November 1993, demonstrates the use of empirical techniques for evaluating uncertainty. The complete paper, which also gives general principles on uncertainty evaluation, is reproduced here for completeness.

The next section derives, from general probability theory, some of the expressions used when treating uncertainty contributions characterised by uniform, or rectangular, distributions. The final expression is used in the example that follows, to determine the standard uncertainties in the input quantities for the measurement problem. Finally, another example is given showing how a level of confidence in an uncertainty interval can be obtained from the number of degrees of freedom.

2 FUNDAMENTAL CONCEPTS

2.1 THE MEASUREMENT PROCESS

Generally, the measurement of a quantity, termed the measurand, is not made directly. Instead, other parameters are measured on which the measurand depends, i.e., the measurand is a function of these other parameters. This can be expressed mathematically as:

\[ Y = f(X_1, X_2, \ldots, X_N) \]  

(1)

where \( Y \) is the measurand and \( X_i \) are the \( N \) other parameters. The \( X_i \) are termed the input quantities, and \( Y \) the output quantity.

For example, the attenuation, \( A \), for a two-port device can be found by measuring the voltage at the input port, \( V_i \), and the voltage at the output port, \( V_o \). In this case, \( V_i \) and \( V_o \) are the input quantities in the

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1 The author is employed by Assessment Services Ltd, and works under contract to the Division of Electrical Science, National Physical Laboratory, based at DRA Malvern.

measurement process and $A$ is the output quantity. The function, $f$, relating the input quantities to the output quantity is:

$$A \ (dB) = 20 \log_{10} \frac{V_i}{V_o} \quad (2)$$

When we measure, we make estimates $(x_1, x_2, \ldots, x_n)$ of the input quantities $(X_1, X_2, \ldots, X_n)$. We can then obtain an estimate, $y$, of the output quantity, $Y$, from our estimates of the input quantities, i.e.:

$$y = f(x_1, x_2, \ldots, x_n) \quad (3)$$

2.2 UNCERTAINTY IN THE INPUT QUANTITIES

To evaluate the uncertainty in the measurand, $Y$, we must first evaluate the uncertainties, $u(x_i)$, in the estimated input quantities, $x_i$. Uncertainty contributions are usually characterised by an estimated standard deviation for the quantity. Such an estimate is termed the standard uncertainty for the quantity. An estimate of the standard deviation for a quantity can be derived either from a series of repeat determinations of the quantity or by other means (such as, from prior experience, or from instrument specifications, data books etc).

Additional consideration should be given if there is dependence between any of the input quantities. This effect, termed correlation, can be avoided by careful consideration during the design stage for the measurement system. (The treatment for correlated input quantities will not be discussed in these notes.)

2.3 UNCERTAINTY IN THE MEASURAND

Having determined estimated values for the input quantities, the result of the measurement can be determined using the functional relationship, $f$. Similarly, the uncertainty in the result of the measurement can be found by propagating the standard uncertainties in the input quantities through the same function to obtain the combined standard uncertainty, $u_c(y)$, in the estimate of measurand, $y$. This can be achieved using empirical techniques, or analytically (by performing partial differentiation on the input quantities). The examples in these notes illustrate the use of both methods of obtaining the combined standard uncertainty in the measurand.

2.4 EXPANDED UNCERTAINTY

The combined standard uncertainty in the measurand is equivalent to an estimate of the standard deviation of the measurand, and can be perceived as an interval containing a specific portion of the population of values attributable to the measurand. (For example, the standard deviation for a Gaussian distribution contains approximately 68% of the total population.) If an interval containing a larger portion of the population is required, the combined standard uncertainty is multiplied by a coverage factor, $k$, to obtain an expanded uncertainty, $U$, in the measurand. i.e.:

$$U = k u_c(y) \quad (4)$$

In most areas of electrical metrology, $k$ is taken as two. This, for a Gaussian distribution, gives an interval which will contain approximately 95% of the population. When quoting an expanded uncertainty for a measurand, it is essential to also give the value of $k$ (or its equivalent in terms of a level of confidence) used to obtain the expanded uncertainty.
3 EXAMPLE 1 – GENERAL PRINCIPLES FOR EVALUATING AND EXPRESSING UNCERTAINTY IN MEASUREMENT

3.1 INTRODUCTION

Measurement attempts to determine the true value of a characteristic for a device. Since all measurement processes are affected by errors, some degree of error will inevitably be present in the measurement result, making it an approximation of the true value. An important part of an overall measurement process is the identification of all the contributing errors and, where possible, making the necessary corrections. Where corrections are not possible, the effect the remaining errors have on the measurement result needs to be evaluated. This is the uncertainty evaluation process for the measurement. The resulting value for the measurement uncertainty indicates quantitatively the doubt about the accuracy of the result. This allows the user of the result to assess its reliability and enables meaningful comparison with other results.

This section of these notes discusses the general principles involved in evaluating and expressing uncertainty in measurement. Each stage of the process is discussed, including identifying, evaluating and combining the uncertainty contributions, and expressing the overall uncertainty. An example is used to illustrate each step of the process. This example concerns a measurement of a nominal 100 Ω resistor made using an impedance bridge operating at 250 MHz. The example is typical and kept simple for clarity.

3.2 UNDERSTANDING THE MEASUREMENT SYSTEM AND IDENTIFYING UNCERTAINTY CONTRIBUTIONS

One of the most important aspects of uncertainty evaluation is the need for a detailed understanding of the overall measurement process. Such understanding is required to ensure all potential contributions to the measurement uncertainty are identified. This means the design engineer or skilled operator of the measurement system is often best suited to perform the evaluation exercise.

The exercise begins by examining in detail the measurement process and identifying potential sources of error. This often involves representing the measurement system using a variety of means, including flow diagrams, computer simulations etc. In our example, the bridge was represented as a block diagram which revealed the following four sources of error; (i) the connection repeatability of the resistor to the bridge, (ii) the electrical noise present on the signals detected by the bridge, (iii) the uncertainty accompanying the transfer standard used to calibrate the bridge, and (iv) the accuracy of the bridge’s frequency source.

Identified errors should be corrected, where possible, e.g., by performing an instrument calibration. Where correction is either not possible or incomplete, the remaining error should be treated as an uncertainty contribution. In our example, all the identified errors remain after calibration, so they are treated as uncertainty contributions.

3.3 QUANTIFYING UNCERTAINTY CONTRIBUTIONS AND THEIR EFFECT ON THE FINAL RESULT

Having identified all the contributions to the uncertainty in the measurement process, each contribution requires evaluation. This means either estimating the limits within which the value for a contribution will be, or estimating the range over which it might vary. The most common statistic used to estimate this range is standard deviation. In our example, the uncertainty associated with the transfer standard (a coaxial capacitor) used to calibrate the bridge can be taken from its calibration certificate, giving us a value of ±0.28 pF. This range, as with the ranges for the other contributions, corresponds to one standard deviation.

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2 Paper presented at the sixth British Electromagnetic Measurements Conference, held at the National Physical Laboratory, Teddington, UK, on 2-4 November 1993.
The frequency source accuracy was found in the manufacturer's specification for the source, giving us a value of ±10 Hz.

The next step is to determine the effect each contribution has on the final measurement result. This can be a difficult process involving a variety of methods, often involving experimentation or based on experience. In our example, the contribution from the transfer standard can be assessed by calibrating the bridge under two conditions; firstly using a value for the transfer standard given on its calibration certificate, then calibrating again using this value plus its uncertainty. The difference between the two subsequent resistance measurements made by the bridge is the uncertainty contribution, which was found to be ±0.26 Ω. The same technique can be used for the contribution from the accuracy of the frequency source — it was found to be ±0.019 Ω. Note that these contributions are in the same units as the final measurement result, i.e., ohms.

Sometimes the two steps given above can be combined, i.e., variations in the contribution can be monitored simultaneously with the effects the variations have on the measurement result. In our example, this can be done for the contribution due to electrical noise by measuring the item repeatedly under essentially the same conditions. Alternatively, the electrical noise and connection repeatability contributions can be evaluated simultaneously using a similar method, but disconnecting and re-connecting the resistor between repeat measurements. The contribution due to connector repeatability and electrical noise, using this technique, was found to be ±0.097 Ω.

3.4 COMBINING THE CONTRIBUTIONS AND EXPRESSING OVERALL UNCERTAINTY

Having identified each contribution and its effect on the measurement result, we need to combine the contributions to give an overall uncertainty figure. In most cases, this can be accomplished by: (i) summing the squared standard deviations of the contributions, (ii) taking the square-root of this sum, and (iii) multiplying the square-root by a coverage factor. The coverage factor is usually chosen to give the required confidence in the result, e.g., for normal distributions, a value of 2 corresponds approximately to a probability of 95% that the uncertainty interval will include the true value (or in other words, a one-in-twenty chance that it won't!). If greater confidence is required, a larger coverage factor can be used, e.g., a value of 3 corresponds approximately to a probability of 99% for the same distribution.

In our example we have;

\[ k \left\{ (0.26)^2 + (0.019)^2 + (0.097)^2 \right\}^{1/2} \Omega \]  

where \( k \) is the coverage factor. If we use \( k = 2 \), then the overall uncertainty for the resistance measurement is ±0.56 Ω.

3.5 DISCUSSION

An uncertainty evaluation process can be extremely complicated or very simple. For measurement systems whose accuracy is governed predominantly by the transfer standard used for its calibration, this can be the only contribution requiring consideration. Our example illustrates this. If we re-calculate the overall uncertainty omitting the uncertainty due to source frequency accuracy, the answer is still ±0.56 Ω. Clearly, this contribution can be ignored. Indeed, if we were to leave out both the frequency accuracy contribution and the electrical noise/connection repeatability contributions, the calculated overall uncertainty would only be reduced to ±0.52 Ω. This is equivalent to assuming all uncertainty in the measurement is due to the uncertainty accompanying the transfer standard used to calibrate the bridge.
The reduced uncertainty produced by omitting contributions can often be compensated for by reducing the number of significant figures used to express the overall uncertainty and rounding the number pessimistically, i.e., by rounding up. This would cause our value of \( \pm 0.52 \, \Omega \) to be rounded up to \( \pm 0.6 \, \Omega \).

Successful identification of components whose contribution to the overall uncertainty is negligible can help reduce considerably the complexity of the uncertainty evaluation process. It is important however, to demonstrate that a given contribution has a negligible effect on a result before omitting it from the calculations.

3.6 SUMMARY

This section discussed some of the more general principles one might encounter during a typical uncertainty evaluation process. Most of the guidelines currently available conform, broadly speaking, to these general principles. Departures inevitably occur, both in written standards and practical aspects of uncertainty evaluation, but these departures are a consequence of the wide variety of situations to which these principles are applied.

4  DERIVING THE STANDARD DEVIATION FOR A UNIFORM, OR RECTANGULAR, DISTRIBUTION

In this section we derive the standard deviation for a uniform, or rectangular, distribution from general probability theory. The standard deviation is also expressed in terms of the limits of the distribution.

4.1  PROBABILITY THEORY

General probability theory defines the mean, \( \bar{x} \), and the variance, \( \sigma^2 \), for any given distribution as:

\[
\bar{x} = \int_{-\infty}^{\infty} x \, p(x) \, dx \quad \text{and} \quad \sigma^2 = \int_{-\infty}^{\infty} (x - \bar{x})^2 \, p(x) \, dx
\]  

(6)

where \( p(x) \) is the probability distribution function.

The standard deviation, \( \sigma \), is the positive square-root of the variance.

4.2  THE RECTANGULAR DISTRIBUTION

For a rectangular distribution, \( p(x) = 1/(2a) \), i.e.;

\[
\begin{array}{c}
\text{---} \\
\text{1/(2a)} \\
\text{---} \\
-a \quad 0 \quad +a \quad x
\end{array}
\]
The mean for this distribution is:

\[ \bar{x} = \int_{-a}^{a} \frac{x}{2a} \, dx = \left[ \frac{x^2}{4a} \right]_{-a}^{a} = \frac{a}{4} - \frac{a}{4} = 0 \]  \hspace{1cm} (7)

and the variance is:

\[ \sigma^2 = \int_{-a}^{a} \left( \frac{x - \bar{x}}{2a} \right)^2 \, dx = \left[ \frac{x^3}{6 \times 2a} \right]_{-a}^{a} = \frac{a^2}{6} + \frac{a^2}{6} = \frac{a^2}{3} \]  \hspace{1cm} (8)

Therefore:

\[ \sigma = \frac{a}{\sqrt{3}} \]  \hspace{1cm} (9)

4.3 STANDARD DEVIATION IN TERMS OF LIMITS

If the limits of the uniform distribution, \( a_+ \) and \( a_- \), are used in place of \( a \), we have:

\[ 2a = a_+ - a_- \hspace{1cm} \text{i.e.} \hspace{1cm} a = (a_+ - a_-)/2 \]  \hspace{1cm} (10)

therefore:

\[ \sigma = \frac{(a_+ - a_-)}{2\sqrt{3}} = \frac{(a_+ - a_-)}{\sqrt{12}} \]  \hspace{1cm} (11)

5 EXAMPLE 2 - DETERMINING THE CHARACTERISTIC IMPEDANCE OF A PRECISION AIR-DIELECTRIC COAXIAL LINE

This example illustrates the evaluation of uncertainty when the measurand is a function of two input quantities. The input quantities are independent and therefore uncorrelated. The example includes treatment of the transformation from one measurand to another, and the effect on the uncertainties.

5.1 DESCRIPTION

Precision air-dielectric coaxial lines are used as standards and verification items for impedance measurements at radio frequencies and above. Their characteristic impedance is a function of the cross-sectional geometry of the outer tube and inner rod which constitute the coaxial line — specifically, the inner diameter of the tube and outer diameter of the rod. Both rod and tube are made of hard metal, usually work-hardened brass.

The lines can be dismantled easily, allowing the inner diameter of the tube and the outer diameter of the rod to be measured using air-gauging systems, which can measure diameters at any point, and orientation, along the rod and tube. The technician operating the system records the maximum and minimum diameters, for both rod and tube, measured by the equipment.

The manufacturer of the air-gauging systems quotes an uncertainty in measurements of both rods and tubes as not greater than 2 μm.
The characteristic impedance, $Z_0$, can be calculated from the diameter measurements of the rod and tube, using:

$$Z_0 = 59.939 \log_e \left( \frac{D}{d} \right)$$

(12)

where $D$ is the inner diameter of the tube and $d$ is the outer diameter of the rod.

5.2 THE CALCULATIONS

The technician's measurements are summarised in the table below. The figures in bold type are used in the calculations, since they represent the maximum and minimum diameters readings including effects caused by the uncertainties in the air-gauging systems.

<table>
<thead>
<tr>
<th>Measurement parameter</th>
<th>Maximum diameter (mm)</th>
<th>Maximum diameter including gauge uncertainty (mm)</th>
<th>Minimum diameter (mm)</th>
<th>Minimum diameter including gauge uncertainty (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inner diameter of the tube</td>
<td>7.007</td>
<td>7.009</td>
<td>6.995</td>
<td>6.993</td>
</tr>
<tr>
<td>Outer diameter of the rod</td>
<td>3.034</td>
<td>3.036</td>
<td>3.030</td>
<td>3.028</td>
</tr>
</tbody>
</table>

5.2.1 Mean values. The mean diameter of the tube, $\overline{D}$, is given by (from equation (10));

$$\overline{D} = \frac{(D_+ + D_-)}{2} = \frac{(7.009 + 6.993)}{2} = 7.001 \ mm$$

(13)

The mean diameter of the rod, $\overline{d}$, is given by (again, from equation (10));

$$\overline{d} = \frac{(d_+ + d_-)}{2} = \frac{(3.036 + 3.028)}{2} = 3.032 \ mm$$

(14)

The mean value for the characteristic impedance is calculated from the two mean diameter values, from equation (12);

$$\overline{Z}_0 = 59.939 \log_e \left( \frac{7.001}{3.032} \right) = 50.16 \ ohms$$

(15)

5.2.2 Uncertainties in the input quantities. The standard uncertainty in the diameter of the tube, $u(D)$, is given by (from equation (11));

$$u(D) = \sqrt{(D_+ - D_-)^2/12} = \sqrt{(7.009 - 6.993)^2/12} = 0.0046 \ mm$$

(16)

The standard uncertainty in the diameter of the rod, $u(d)$, is given by (again, from equation (11));

$$u(d) = \sqrt{(d_+ - d_-)^2/12} = \sqrt{(3.036 - 3.028)^2/12} = 0.0023 \ mm$$

(17)
5.2.3 Uncertainty in the measurand. The combined standard uncertainty in the measurand, \( u_c(y) \), can be found using the following equation;

\[
u_c^2(y) = \sum_{i=1}^{N} \left( \frac{\partial f}{\partial x_i} \right)^2 u^2(x_i)
\]  

(18)

which combines the standard uncertainties in the input quantities (the diameter measurements) to give the standard uncertainty in the measurand (the characteristic impedance).

First calculate the partial derivatives of \( Z_0 \) with respect to \( D \) and \( d \), using equation (12):

\[
\frac{\partial Z_0}{\partial D} = \frac{59.939}{D} \quad ; \quad \frac{\partial Z_0}{\partial d} = -\frac{59.939}{d}
\]

(19)

\[\text{i.e.}\]

\[
u_c^2(Z_0) = \sum_{i=1}^{2} \left( \frac{\partial Z_0}{\partial x_i} \right)^2 u^2(x_i)
\]

(20)

\[
\begin{align*}
\left( \frac{\partial Z_0}{\partial D} \right)^2 u^2(D) + \left( \frac{\partial Z_0}{\partial d} \right)^2 u^2(d) = \left( \frac{59.939}{D} \right)^2 u^2(D) + \left( \frac{59.939}{d} \right)^2 u^2(d)
\end{align*}
\]

(21)

Therefore

\[
u_c^2(Z_0) = \left( \frac{59.939}{7.001} \right)^2 (0.0046)^2 + \left( \frac{59.939}{3.032} \right)^2 (0.0023)^2 = 0.0036
\]

(22)

\[
u_c(Z_0) = 0.060 \ \text{ohms}
\]

(23)

Or, for an expanded uncertainty using a coverage factor of two (from equation (4));

\[
U = k u_c(Z_0) = 0.12 \ \text{ohms} \quad (\text{for} \ k = 2)
\]

(24)

5.3 TRANSFORMATION TO REFLECTION COEFFICIENT

Some customers for these measurements request that the magnitude of the complex voltage reflection coefficient for the coaxial line is given. The magnitude of the complex voltage reflection coefficient, \(|\Gamma|\), with respect to 50 ohms, is related to the characteristic impedance by the following bi-linear transformation;

\[
|\Gamma| = \left| \frac{Z_0 - 50}{Z_0 + 50} \right|
\]

(25)

5.3.1 Mean value. The mean value for the magnitude of the complex voltage reflection coefficient is simply given by using the mean value of \( Z_0 \) in equation (25);

\[
|\bar{\Gamma}| = \left| \frac{50.16 - 50}{50.16 + 50} \right| = 0.0016
\]

(26)
5.3.2 Uncertainty in the measurand. Equation (18) is used to transform the standard uncertainty in \(Z_0\) into a standard uncertainty in \(|\Gamma|\).

First calculate the partial derivative of \(|\Gamma|\) with respect to \(Z_0\), using equation (25):

\[
\frac{\partial |\Gamma|}{\partial Z_0} = \frac{100}{(Z_0 + 50)^2}
\]  
(27)

Then use equation (18) to transform the standard uncertainty in \(Z_0\) into a standard uncertainty in \(|\Gamma|\):

\[
u_c^2(|\Gamma|) = \left(\frac{\partial |\Gamma|}{\partial Z_0}\right)^2 u^2(Z_0)
\]  
(28)

Therefore,

\[
u_c^2(|\Gamma|) = \left(\frac{100}{(50.16 + 50)^2}\right)^2 0.060^2 = 3.58 \times 10^{-7}
\]  
(29)

\[
u_c(|\Gamma|) = 0.00060
\]  
(30)

Or, for an expanded uncertainty using a coverage factor of two (from equation (4));

\[
U = k u_c(|\Gamma|) = 0.0012 \quad (\text{for } k = 2)
\]  
(31)

6 **EXAMPLE 3 – A SPECIAL CASE INVOLVING DEGREES OF FREEDOM**

This example shows how a level of confidence in an uncertainty interval can be obtained from the number of degrees of freedom. The example is kept simple for clarity.

6.1 DESCRIPTION

The transmission coefficient at 8 GHz of a reciprocal two-port device is measured using a microwave network analyser calibrated using the TRL technique. The device is measured six times, re-calibrating the network analyser before each re-measurement. Such measurements are dominated by random errors, so any systematic errors have been ignored.

6.2 THE MEASUREMENTS

The six transmission coefficient measurements, \(\tau_i\), are given below;

\(\tau_i\; 0.674, \; 0.658, \; 0.667, \; 0.691, \; 0.654 \; \text{and} \; 0.672\)
6.3 THE CALCULATIONS

First calculate the arithmetic mean of the observations, $\bar{\tau}$, using the following equation:

$$\bar{\tau} = \frac{1}{n} \sum_{i=1}^{n} \tau_i = \frac{1}{6} (0.674 + 0.658 + 0.667 + 0.691 + 0.654 + 0.672) = 0.6693$$

(32)

Then calculate the estimated standard deviation, $s(\tau)$, as follows:

$$s(\tau) = \sqrt{\left( \frac{\sum_{i=1}^{n} (\tau_i - \bar{\tau})^2}{n - 1} \right)} = \sqrt{\left( \frac{8.673 \times 10^{-4}}{6 - 1} \right)} = 0.01317$$

(33)

and finally the standard uncertainty, $u(\tau)$;

$$u(\tau) = \frac{s(\tau)}{\sqrt{n}} = \frac{0.01317}{\sqrt{6}} = 0.00538$$

(34)

Now, the number of degrees of freedom, in this case, is given by $(n - 1)$;

$$6 - 1 = 5$$

(35)

6.3.1 95% level of confidence. If a 95% level of confidence is required, we look up the appropriate value for $t$ for 5 degrees of freedom, in the 95% column of a $t_p(v)$ table (e.g., see the appendix);

$$t_{95\%}(5) = 2.57$$

(36)

i.e., $k_p = k_{95\%} = t_{95\%}(5) = 2.57$

(37)

$$U_{95\%} = k_{95\%} \times u(\tau) = 2.57 \times 0.00538 = 0.014$$

(38)

Therefore, the transmission coefficient measurement is expressed as;

$$\tau = 0.669 \pm 0.014$$

(39)

where the uncertainty quoted is an expanded uncertainty at a 95% level of confidence.

6.3.2 99.7% level of confidence ($3\sigma$). If a 99.7% level of confidence is required (corresponding to three standard deviations, or $3\sigma$), we look up the appropriate value for $t$ for 5 degrees of freedom, in the 99.7% column of a $t_p(v)$ table (see the appendix);

$$t_{99.7\%}(5) = 5.51$$

(40)

i.e., $k_p = k_{99.7\%} = t_{99.7\%}(5) = 5.51$

(41)

$$U_{99.7\%} = k_{99.7\%} \times u(\tau) = 5.51 \times 0.00538 = 0.030$$

(42)

Therefore, the transmission coefficient measurement is expressed as;

$$\tau = 0.669 \pm 0.030$$

(43)

where the uncertainty quoted is an expanded uncertainty at a 99.7% level of confidence.
7 CONCLUSIONS

These notes have demonstrated evaluating uncertainty in a number of situations using different techniques. Other situations encountered in metrology may be dealt with using these methods or, more frequently, a combination of the methods. For a fuller treatment of this subject the reader is advised to consult the ISO Guide [1], or, alternately, some of the excellent documents which have been written based on the Guide for different applications. For example, in the UK, NAMAS have published documents specific to calibration [2], testing [3] and EMC measurements [4].

Work is continuing in this field at both national and international levels, the result of which will be of benefit to practitioners, customers for measurements, and suppliers.

8 REFERENCES


APPENDIX

TABLE OF VALUES FOR \( t_\nu(p) \)

This appendix contains a table of values for \( t_\nu(p) \), often called Student’s-t, after the originator, W.L. Gossett, who used the nom de plume of ‘Student’. Three columns of values are given for levels of confidence; 68.3%, 95.0% and 99.7%, corresponding approximately to one, two and three standard deviations of a normal, or Gaussian, distribution.

<table>
<thead>
<tr>
<th>Degrees of freedom</th>
<th>Fraction ( p ) in percent (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \nu )</td>
<td>68.3</td>
</tr>
<tr>
<td>1</td>
<td>1.84</td>
</tr>
<tr>
<td>2</td>
<td>1.32</td>
</tr>
<tr>
<td>3</td>
<td>1.20</td>
</tr>
<tr>
<td>4</td>
<td>1.14</td>
</tr>
<tr>
<td>5</td>
<td>1.11</td>
</tr>
<tr>
<td>6</td>
<td>1.09</td>
</tr>
<tr>
<td>7</td>
<td>1.08</td>
</tr>
<tr>
<td>8</td>
<td>1.07</td>
</tr>
<tr>
<td>9</td>
<td>1.06</td>
</tr>
<tr>
<td>( \infty )</td>
<td>1.00</td>
</tr>
</tbody>
</table>