Using data visualisation techniques to explore the random error distribution of two-port VNA measurements

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Abstract

A method for visualising the complex valued S-parameter data for 2-port microwave devices is presented. It consists of plotting the value and an elliptical uncertainty region in the plane at each frequency for each S-parameter. Some measured S-parameter data is presented in this way.

Introduction

The S-parameters of a two-port microwave device as measured by a vector network analyser (VNA) are complex valued - they have magnitude and phase and so are inherently two dimensional. Each S-parameter is represented by a point in the plane and its uncertainty is represented by a region in the plane centred on the point.

The quoted value and uncertainty of each S-parameter are actually based on a statistical analysis of \( n \) repeat measurements (where \( n \) is typically in the range 2 to 10). The device is disconnected, the VNA is recalibrated and the device is reconnected between each pair of measurements. A mean value and standard deviation in the mean are specified for both components of the parameter as is the correlation coefficient between the two components. Thus for each S-parameter there are five numbers at each frequency. There are four S-parameters and perhaps several hundred frequencies. It is extremely useful to be able to pick out trends and correlations in the data but this is difficult to accomplish without visual presentation.

In this paper a technique for visualising the data is discussed which consists of plotting an elliptical uncertainty region at each frequency for the four S-parameters of a 2-port device and watching the ellipses evolve with frequency. The ellipse is centred on the mean complex value and its size, shape and orientation are determined by the standard deviation in the means of the two components and the correlation coefficient between them.

Sample statistics

Due to random effects such as receiver noise and connector repeatability successive measurements of an S-parameter (with disconnection, recalibration and reconnection as described above) yield slightly different values. The values obtained can be modelled as a complex valued (or 2-dimensional vector valued) random variable distributed according to some bivariate (i.e. two-dimensional) distribution. The parameters of the distribution are determined by the device, the measurement system and the engineer performing the measurements. Thus a set of \( n \) repeat measurements can be regarded as a random sample of size \( n \) from this bivariate distribution.

If the sample consists of the \( n \) complex numbers \( x_i + j y_i \) (\( i = 1 \) to \( n \)) then the means of the real and imaginary parts (\( \bar{x} \) and \( \bar{y} \)) are given by
\[ x = \frac{1}{n} \sum_{i=1}^{n} x_i \]
\[ y = \frac{1}{n} \sum_{i=1}^{n} y_i \]

If several sets of repeat measurements (i.e. several samples) were taken then the mean values obtained would vary from sample to sample. The uncertainty of the mean values is measured by the standard deviations in the mean of the real and imaginary parts (\( s(\bar{x}) \) and \( s(\bar{y}) \)) which are given by:

\[ s(\bar{x}) = \sqrt{s^2(\bar{x})} \]
\[ s(\bar{y}) = \sqrt{s^2(\bar{y})} \]

where the variances in the mean of the real and imaginary parts (\( s^2(\bar{x}) \) and \( s^2(\bar{y}) \)) are given by:

\[ s^2(\bar{x}) = \frac{1}{n(n-1)} \sum_{i=1}^{n} (x_i - \bar{x})^2 \]
\[ s^2(\bar{y}) = \frac{1}{n(n-1)} \sum_{i=1}^{n} (y_i - \bar{y})^2 \]

Correlation can be broadly defined as the degree to which two parameters are interrelated. For example if some physical phenomenon (e.g. discontinuities in a connector) causes both the real and imaginary parts of the reflection coefficient of a device to increase then the real and imaginary parts are said to be positively correlated. Conversely, if one component increases but the other decreases then negative correlation is present. The degree of correlation between two measured components can be measured using the correlation coefficient \( r(\bar{x}, \bar{y}) \) where

\[ r(\bar{x}, \bar{y}) = \frac{s(\bar{x}, \bar{y})}{\sqrt{s^2(\bar{x}) s^2(\bar{y})}} = \frac{s(\bar{x}, \bar{y})}{s(\bar{x}) s(\bar{y})} \]

where the covariance in the mean between real and imaginary parts is defined to be:

\[ s(\bar{x}, \bar{y}) = \frac{1}{n(n-1)} \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y}) \]

The correlation coefficient varies between -1 and +1. It is positive if \( \bar{x} \) and \( \bar{y} \) are positively correlated, zero if they are uncorrelated and negative if they are negatively correlated.

The covariance matrix in the mean is

\[
\begin{pmatrix}
  s^2(\bar{x}) & s(\bar{x}, \bar{y}) \\
  s(\bar{x}, \bar{y}) & s^2(\bar{y})
\end{pmatrix}
= \begin{pmatrix}
  s^2(\bar{x}) & r(\bar{x}, \bar{y}) \sqrt{s^2(\bar{x}) s^2(\bar{y})} \\
  r(\bar{x}, \bar{y}) \sqrt{s^2(\bar{x}) s^2(\bar{y})} & s^2(\bar{y})
\end{pmatrix}
\]

Note that \( s(\bar{x}, \bar{y}) = s(\bar{y}, \bar{x}) \) so that the covariance matrix is symmetric.

**Bivariate normal distributions and elliptical uncertainty regions**

Here it will be assumed that the complex random variable representing the repeat measurements is distributed according to a bivariate normal distribution. Consider the case when the real part \( x \) and the imaginary part \( y \) of an S-parameter \( x+jy \) are independently normally distributed with means \( \mu_x \), \( \mu_y \) and variances \( \sigma_x^2 \), \( \sigma_y^2 \) respectively. Then the probability density function for the real part is proportional to
\[ \exp \left( -\frac{(x-\mu_x)^2}{2\sigma_x^2} \right) \]

and the probability density function for the imaginary part is proportional to

\[ \exp \left( -\frac{(y-\mu_y)^2}{2\sigma_y^2} \right) \]

Hence the joint (bivariate) probability density function is proportional to

\[ \exp \left( -\frac{(x-\mu_x)^2}{2\sigma_x^2} - \frac{(y-\mu_y)^2}{2\sigma_y^2} \right) \]

The exponent in the above expression can be rewritten as

\[ -\frac{1}{2} e^T V^{-1} e \]

where the vector \( e \) is given by

\[ e = \begin{pmatrix} x-\mu_x \\ y-\mu_y \end{pmatrix} \]

and the matrix \( V \) is given by

\[ \begin{pmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{pmatrix} \]

The positive definite, diagonal matrix \( V \) is the covariance matrix between the real and imaginary parts in the case when they are uncorrelated (independent). More generally in the case when the real and imaginary parts are correlated the bivariate probability density function takes the same form involving the covariance matrix. The covariance matrix is no longer diagonal but is still symmetric and positive definite.

In the more general case the probability contours have the equation

\[ \begin{pmatrix} x-\mu_x \\ y-\mu_y \end{pmatrix} \begin{pmatrix} \frac{1}{(1-\rho^2)\sigma_x^2} & -\frac{\rho}{(1-\rho^2)\sigma_x\sigma_y} \\ -\frac{\rho}{(1-\rho^2)\sigma_x\sigma_y} & \frac{1}{(1-\rho^2)\sigma_y^2} \end{pmatrix} \begin{pmatrix} x-\mu_x \\ y-\mu_y \end{pmatrix} = k^2 \]

where \( \rho \) is the correlation coefficient between the two components, \( k \) is a coverage factor and the matrix is the inverse of the covariance matrix. This can be written as

\[ \frac{(x-\mu_x)^2}{\sigma_x^2} - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} + \frac{(y-\mu_y)^2}{\sigma_y^2} = k^2(1-\rho^2) \]

which represents an ellipse. Thus the probability contours for a general bivariate distribution are elliptical. If \( k \) is chosen to be 2.5 then the probability that a point \( x+iy \) chosen at random will lie inside the corresponding ellipse is approximately 95%. Note that the coverage factor for a 95% confidence region in two dimensions is 2.5 whereas the corresponding coverage factor in one dimension for a 95% confidence interval is 2.

If individual values are normally distributed with mean \( (\mu_x, \mu_y) \), variance \( (\sigma_x^2, \sigma_y^2) \) and covariance \( \sigma_{xy} \), then sample mean is normally distributed with mean \( (\mu_x, \mu_y) \), variance \( \left( \frac{\sigma_x^2}{n}, \frac{\sigma_y^2}{n} \right) \) and covariance \( \frac{\sigma_{xy}}{n} \)
Animated ellipses

The 2-port devices measured were a 20 dB attenuator, a 40 dB attenuator, a Beatty standard (a 50 ohm airline containing a 25 ohm section) and a matched 50 ohm airline. The devices were all fitted with GPC3.5 connectors. The S-parameters of the four devices were measured at 0.2 GHz intervals over the frequency range 0.2 - 33 GHz. Since it is the orientation and relative size of the ellipses which is of most interest the ellipses were scaled to make them visible. The scale factor varies from one frequency to the next. All four ellipses were scaled by the same factor to make the ellipses for the transmission parameters visible. As a result the ellipses for the reflection parameters are sometimes off the scale of the picture. Some sample results are shown in Figs 1-4. In the figures parameters 1 and 4 are reflection parameters (S11 and S22 respectively) whilst parameters 2 and 3 are transmission parameters (S12 and S21 respectively). For each device the sequence of pictures was turned into an animation. It is noticeable from the animations that the ellipses for the two transmission parameters tend to track one another fairly closely but that ellipses for the two reflection parameters are often quite different.

Conclusion

S-parameter measurements on 2-port devices at a large number of frequencies generate large data sets. A visual representation of the data makes it much easier to see the frequency dependence of the device and to spot trends and correlation in its behaviour.

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Fig 1 Expanded uncertainty regions for S-parameters of 20 dB attenuator at (from right to left) 0.2, 13, 25, 33 GHz

Fig 2 Expanded uncertainty regions for S-parameters of 40 dB attenuator at (from right to left) 0.2, 13, 25, 33 GHz
Fig 3 Expanded uncertainty regions for S-parameters of Beatty standard at (from right to left) 0.2, 13, 25, 33 GHz

Fig 4 Expanded uncertainty regions for S-parameters of matched air line at (from right to left) 0.2, 13, 25, 33 GHz